THE SEPARATION CONDITION OF BOUNDARY LAYERS

K. Oswatitsch

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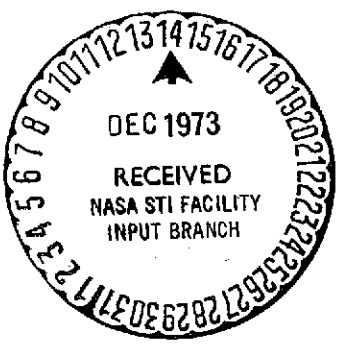

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16. Abstract: The author deals with the conditions under which a three-dimensional flow detaches itself from the wall. The prerequisites are broadly conceived; it is merely assumed that the velocity vector near the wall can be adequately approximated in the vicinity of the separation point by means of the beginning of a Taylor series. Applying the continuity equation, Navier-Stokes equations and the no-slip condition, one finds that the Taylor coefficients up to second-order terms, including the point of separation, can be expressed by components τ and σ of the wall shearing stress, their derivatives, and the derivatives of p along the wall. By this means, a system of ordinary differential equations is obtained for the flow lines of a viscous flow near the wall, the solutions of which result in a complete survey of the local separation attitude. Only the wall shearing stress distribution is responsible for the type of flow; for strength and direction of the separation, the pressure gradient is also a determining factor. The separation line in the three-dimensional case is also the wall flow line; this is not so in the two-dimensional case. In addition, the differentiation between "separation" and "deflection" in the three-dimensional case is made.

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THE SEPARATION CONDITION OF BOUNDARY LAYERS

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1. Introduction

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The research pertains to the conditions under which a three-dimensional flow detaches itself from the wall.

The prerequisites are as broadly conceived as possible. The wall may be curved, the flow compressible, thermal conductivity existing and the Reynolds number arbitrary, as long as flow is stationary. It is merely assumed that the velocity vector near the wall can also be adequately approximated in the vicinity of the separation point by means of the beginning of a Taylor series.[1]

By means of applying the continuity equation, the Navier-Stokes equations (that is, NOT the boundary layer equations) and the no-slip condition, it turns out that the Taylor coefficients up to the terms of the second order, including the point of separation, can be expressed by the components τ and σ of the wall shearing stress and their derivatives and the derivatives of p along the wall. By this means, a system of ordinary differential equations is obtained for the flow lines of a viscous flow near the wall, the solutions of which can be taken in at a glance and that result in a complete survey of the local separation attitude. For the <u>type</u> of flow, only the wall shearing stress distribution is responsible; for the strength and direction of the separation, however, the pressure gradient is also a determining factor.

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*Numbers in the margin indicate pagination in the foreign text.

[1] A special singular behavior is treated in [1] for the two-dimensional case.

The three-dimensional separation differs from the two-dimensional one to a significant extent inasmuch as the components of the wall shearing stress along the "separation line," which divides the region of oncoming flow and return flow, disappear in identical measure in the two-dimensional and rotationally symmetric case, whereas this happens only at isolated points in the general three-dimensional flow, if a separation line exists at all. In general, the separation line in the three-dimensional case is also the wall flow line, /358
while this is not so in the two-dimensional case. In addition, it is useful to differentiate in the three-dimensional case between "separation" and "deflection." There is separation if a wall flow line finds a continuation in a separation flow line leading away from the wall, deflection if it only branches off at the wall, thereby bypassing a region of return flow. The disappearance of wall shearing stress proves to be a necessary but not an adequate prerequisite for separation at a point.

2. The Course of Velocity, Shearing Stress and Pressure at the Wall

In the following, x, y, z are the Cartesian coordinates, the y-axis of which is normal to the wall presumed to be level. (Exactly corresponding considerations lead to identical results for curved walls.) The direction of x and z is not tied in with the potential flow.

The velocity vector $w = w(u,v,w)$ disappears on account of $y = 0$. The same holds true for an arbitrary $m, n \geq 0$.

$$\frac{\partial^{m+n} w}{\partial x^m \partial z^n} = 0. \qquad (1)$$

By the addition of the continuity condition for compressible media of density ρ

(2)

it follows that: $\operatorname{div} w + w \operatorname{grad} (\ln \varrho) = 0$

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} = \left.\frac{\partial^2 v}{\partial x \partial y}\right|_{y=0} = \left.\frac{\partial^2 v}{\partial y \partial z}\right|_{y=0} = \cdots = 0. \tag{3}$$

Differentiating (2) with respect to y with (1) and (3) results in the following for y = 0:

$$-\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial u}{\partial y} \cdot \frac{1}{\varrho} \frac{\partial \varrho}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{1}{\varrho} \frac{\partial \varrho}{\partial z}. \tag{4}$$

If τ is the component of the wall shearing stress in the x, σ the one in the z direction, μ the coefficient of friction, then in accordance with (1) at y = 0 we get:

$$\tau = \mu \cdot \frac{\partial u}{\partial y}, \quad \sigma = \mu \cdot \frac{\partial w}{\partial y}. \tag{5}$$

Assuming that μ depends only on the temperature T(x,y,z), as is the case with ideal gases and the common fluids, then for y = 0, (5) results in:

$$\left.\begin{aligned} \tau_x &= \frac{\mu_x}{\mu} \cdot \tau + \mu \cdot \frac{\partial^2 u}{\partial x \partial y}; & \tau_z &= \frac{\mu_z}{\mu} \cdot \tau + \mu \cdot \frac{\partial^2 u}{\partial y \partial z}; \\ \sigma_x &= \frac{\mu_x}{\mu} \cdot \sigma + \mu \cdot \frac{\partial^2 w}{\partial x \partial y}; & \sigma_z &= \frac{\mu_z}{\mu} \cdot \sigma + \mu \cdot \frac{\partial^2 w}{\partial y \partial z} \end{aligned}\right\} \tag{6}$$

while by the use of the complete Navier-Stokes equations (not by simplified forms of them arrived at by boundary layer assumptions), the following also results for y = 0: /359

$$\frac{\partial p}{\partial x} = \frac{\mu_y}{\mu} \cdot \tau + \mu \frac{\partial^2 u}{\partial y^2}; \quad \frac{\partial p}{\partial z} = \frac{\mu_y}{\mu} \sigma + \mu \cdot \frac{\partial^2 w}{\partial y^2}. \tag{7}$$

By this means, all second derivatives of u, v, w can be suitably represented; by the use of (6) and (7), (4) can be written as follows:

$$-\mu \cdot \frac{\partial^2 v}{\partial y^2} = \tau_x + \sigma_z - \frac{\mu_x}{\mu}\tau - \frac{\mu_z}{\mu}\sigma + \frac{\tau}{\varrho}\frac{\partial \varrho}{\partial x} + \frac{\sigma}{\varrho}\frac{\partial \varrho}{\partial z}. \qquad (8)$$

On account of the no-slip conditions (1), the Taylor series for w in the region of the wall point x = y = z = 0

$$w = w_y \cdot y + w_{xy} \cdot xy + 1/2\, w_{yy} \cdot y^2 + w_{yz} \cdot yz + \dots \qquad (9)$$

The vector derivatives to be taken at x = y = z = 0 have the following components, according to (5) and (8):

w	u	v	w
μw_y	τ ;	0 ;	σ ;
μw_{xy}	$\tau_x - \frac{\mu_x}{\mu}\tau$;	0 ;	$\sigma_x - \frac{\mu_x}{\mu}\sigma$;
μw_{yy}	$p_x - \frac{\mu_y}{\mu}\tau$;	$-\left[\tau_x + \sigma_z - \frac{\mu_x}{\mu}\tau - \frac{\mu_z}{\mu}\sigma + \frac{\tau}{\varrho}\frac{\partial \varrho}{\partial x} + \frac{\sigma}{\varrho}\frac{\partial \varrho}{\partial z}\right]$;	$p_z - \frac{\mu_y}{\mu}\sigma$;
μw_{yz}	$\tau_z - \frac{\mu_z}{\mu}\tau$;	0 ;	$\sigma_z - \frac{\mu_z}{\mu}\sigma$.

(10)

It turns out that at a point $\tau = \sigma = 0$, which in the following becomes highly interesting, the effect of compressibility as well as the effect of a three-dimensionally varying temperature drop out. With this, the course of flow lines in the vicinity of a separation point, in the framework of approximation (9), is also traced back to the pressure and shearing stress conditions at the wall.

3. Separation During Two-Dimensional Flow

With a two-dimensional or axially symmetrical flow (see [2]), the coordinate system can be arranged so that $w = \sigma = 0$. At the same time, the z-dependency of the other quantities

disappears. This results in:

$$\left.\begin{aligned} \mu \cdot u &= \tau \cdot y + \left(\tau_x - \frac{\mu_x}{\mu}\tau\right) x\, y + \frac{1}{2}\left(p_x - \frac{\mu_y}{\mu}\tau\right) y^2, \\ \mu \cdot v &= -\frac{1}{2}\left(\tau_x - \frac{\mu_x}{\mu}\tau + \frac{\tau}{\varrho}\varrho_x\right) y^2 \end{aligned}\right| \qquad (11)$$

and

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$$\frac{dy}{dx} = \frac{v}{u} = \frac{-\left(\tau_x - \frac{\mu_x}{\mu}\tau + \frac{\tau}{\varrho}\varrho_x\right) y}{2\tau + 2\left(\tau_x - \frac{\mu_x}{\mu}\cdot\tau\right) x + \left(p_x - \frac{\mu_y}{\mu}\tau\right) y}. \qquad (12)$$

It is evident from (12) that a flow separation line ($\tan\theta \neq 0$) at $x = y = 0$ can only occur for $\tau = 0$. This results in:

$$\frac{dy}{dx} = \frac{-\tau_x \cdot y}{2\tau_x \cdot x + p_x \cdot y}. \qquad (13)$$

Conditions for this equation are met by wall flow line $y = 0$ and flow separation line $y = x \cdot \tan\theta$;

$$\operatorname{tg}\vartheta = -\frac{3\tau_x}{p_x}. \qquad (14)$$

θ is the angle of separation. The greater the speed with which the wall shearing stress decreases and the smaller (!) the increase in pressure, the greater the angle. The separation angle θ can be equal to 0 if τ_x also = 0, in spite of $p_x = 0$. The Hartree separation profile is an example. The general solution of (13) is:

$$y^3 (x - y \cot\vartheta) = \text{const.}. \qquad (15)$$

Figure 1 shows the flow lines corresponding to (14) and (15) and also the curve of the disappearing u-component

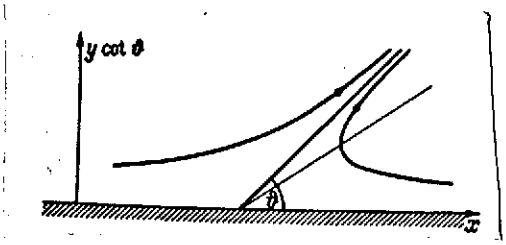

Fig. 1. Point of separation for two-dimensional flow. Additional flow lines follow from the ones plotted by similarity transformations with the point of separation as center.

$y = 2/3 \tan\theta \cdot x$. It is already contained in the classic work of Prandtl [3]. Equation (14) and its generalization for three dimensions (22) are in essence already contained in R. Legendre [4].

The question is still open regarding the distance from the wall up to which the solution is veracious. Measurements by E.A. Eichelbrenner and H. Werlé [5] show that in certain cases the flow separation line, following an intial strong slope, quickly deflects in the direction of the wall. Closer consideration is also merited by the case of a supersonic flow at a concave wall, where it is not possible to experimentally demonstrate separation in spite of the highest pressure increase, whereas the hitherto known theories of boundary layers show a separation already at the onset of compression at a completely finite angle of separation.

The experimental proof of (14) is usually out of the question, on account of the difficulty of measuring τ, whereas a theoretical determination of τ_x is based on the prerequisite of methds of calculation for boundary layers that also function at the point of separation. Because the following results from the second Navier-Stokes equation with (8) at the point of separation: /361

$$p_y = \mu \cdot \frac{\partial^2 v}{\partial y^2} = -\tau_x = \frac{1}{3} \operatorname{tg} \vartheta \cdot p_x, \qquad (16)$$

with a "strong" separation, a failure of the boundary approximation $p_y < p_x$ must be reckoned with.

4. The Symmetrical Three-Dimensional Point of Separation

A simple form of separation of a three-dimensional boundary layer results from a flow symmetrical to the plane z = 0, as it might occur at the top or bottom of a wing cross section of finite aspect ratio. In that case, the following holds true:

$$\left.\begin{aligned} \tau(x,y,z) &= \tau(x,y,-z); \qquad p(x,y,z) = p(x,y,-z); \\ \sigma(x,y,z) &= -\sigma(x,y,z) \end{aligned}\right| \tag{17}$$

and in consequence:

$$\left.\tau_z(x,y,0)=0;\quad p_z(x,y,0)=0;\quad \sigma(x,y,0)=0;\quad \sigma_x(x,y,0)=0.\right| \tag{18}$$

Velocity distribution is then reduced to the representation:

$$\left.\begin{aligned} \mu\cdot u &= \tau\cdot y+\left(\tau_x-\frac{\mu_x}{\mu}\tau\right)xy+\frac{1}{2}\left(p_x-\frac{\mu_y}{\mu}\tau\right)y^2; \\ \mu\cdot v &= -\frac{1}{2}\left(\tau_x+\sigma_z-\frac{\mu_x}{\mu}\tau+\frac{\tau}{p}p_x\right)y^2; \\ \mu\cdot w &= \sigma_z\cdot yz. \end{aligned}\right| \tag{19}$$

The flow lines are represented by:

$$\left.dx:dy:dz=u:v:w.\right| \tag{20}$$

By forming dy/dx, the fact is determined as in (12) that $\tau = 0$ is a necessary prerequisite for a flow line slope differing from zero. The entire wall shearing stress again disappears in a point on account of $\sigma = 0$.

If y is selected as an independent variable, (19) and (20) yield the following for the vicinity of the point of separation:

$$\left.\begin{aligned} \frac{u}{v}=\frac{dx}{dy}&=\frac{2\tau_x\cdot x+p_x\cdot y}{-(\tau_x+\sigma_z)y}, \\ \frac{w}{v}=\frac{dz}{dy}&=\frac{2\sigma_z\cdot z}{-(\tau_x+\sigma_z)y}. \end{aligned}\right| \tag{21}$$

The following equation for a flow separation line results from a generalization of (14) $y = x \cdot \tan\theta$, $z = 0$, with

$$\operatorname{tg}\vartheta = -\frac{\delta\tau_x + \sigma_z}{p_x}. \qquad (22)$$

By means of the affine transformation /362

$$\xi = x - y\cot\vartheta, \quad \eta = y, \quad \zeta = z \qquad (23)$$

(21) receives the normalized form of

$$\frac{d\xi}{dy} = -\frac{2\tau_x}{\tau_x + \sigma_z}\cdot\frac{\xi}{y}; \quad \frac{dz}{dy} = -\frac{2\sigma_z}{\tau_x + \sigma_z}\cdot\frac{z}{y} \qquad (24)$$

with solution

$$\begin{aligned} y^{\frac{2\tau_x}{\tau_x+\sigma_z}}\cdot\xi &= y^{\frac{2\tau_x}{\tau_x+\sigma_z}}\cdot(x - y\cot\vartheta) = C_1, \\ y^{\frac{2\sigma_z}{\tau_x+\sigma_z}}\cdot z &= C_2. \end{aligned} \qquad (25)$$

For the wall flow lines, we obtain

$$x^{-\frac{\sigma_z}{\tau_x}}\cdot z = C. \qquad (26)$$

If matter flows toward or away from the x-axis, the term referred to is convergence or divergence point, and $\sigma_z < 0$ or $\sigma_z > 0$. It is appropriate to speak of weak convergence for $0 < \sigma_z/\tau_x < 1$ and of strong convergence for $\sigma_z/\tau_x > 1$. Analogously, weak and strong divergence is differentiated for $-1/3 < \sigma_z/\tau_x < 0$ and $\sigma_z/\tau_x < -1/3$. Two-dimensional flow corresponds to the value of $\sigma_z/\tau_x = 0$.

The following table offers a compilation of characteristic symmetrical separation centers.

Figures 2-9 in the text beneath the illustrations represent the cases as noted in the table.

Case	σ_z	τ_x; $-p_x$	σ_z/τ_x	$\frac{2\tau_x}{\tau_x+\sigma_z}$	$\frac{2\sigma_z}{\tau_x+\sigma_z}$	$1+\frac{1}{3}\frac{\sigma_z}{\tau_x}$	
1	<0	>0	-4	$-2/3$	$8/3$	$-1/3$	Convergence ↑ strong
2	<0	>0	-3	-1	3	0	
3	<0	<0	2	2/3	4/3	5/3	
4	<0	<0	1	1	1	4/3	—
5	<0	<0	1/2	4/3	2/3	7/6	weak
6	0	<0	0	2	0	1	Two-dimensional flow
7	>0	<0	$-1/5$	5/2	$-1/2$	14/15	weak
8	>0	<0	$-1/3$	3	-1	8/9	—
9	>0	<0	$-1/2$	4	-2	5/6	strong ↓
10	>0	<0	-1	∞	$-\infty$	2/3	Divergence No separation
11	>0	<0	-2	-2	$+4$	1/3	

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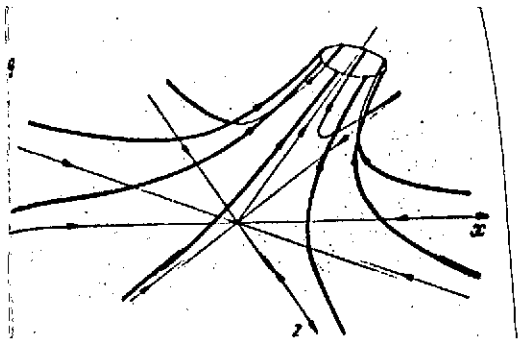

Fig. 2. Point of convergence of average convergence (Case 4 of table). Flow tends radially toward the separation center from the onflow as well as from the return-flow side. Straight wall flow lines that continue in the flow separation line.

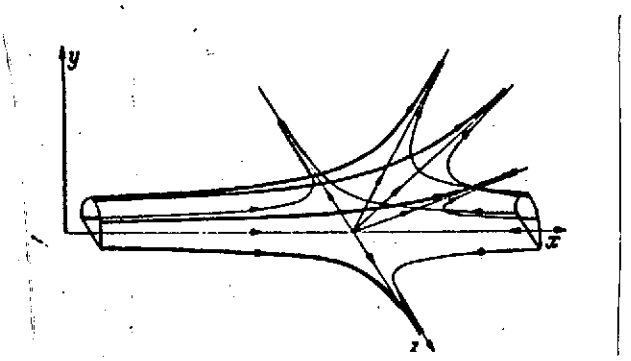

Fig. 3. Point of divergence of average divergence (Case 8 in table). Two flow planes of semicircular cross section, tending toward the separation plane $x = y\cot\theta$ from the on-flow and the return-flow side.

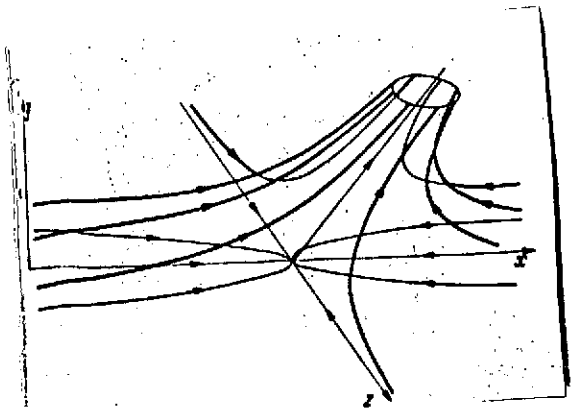

Fig. 4. Separation center for weak convergence (Case 5 of table).

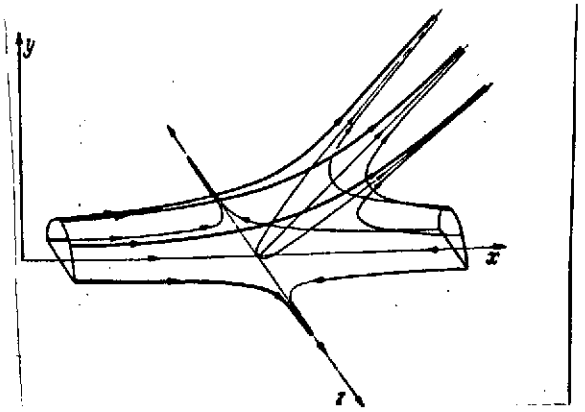

Fig. 5. Point of divergence of weak divergence (Case 7).

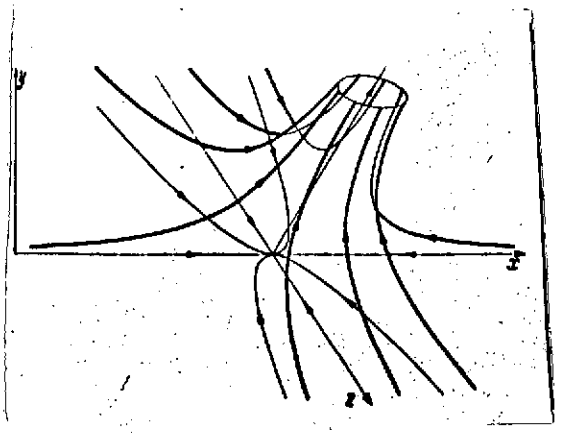

Fig. 6. Separation center for strong convergence (Case 3).

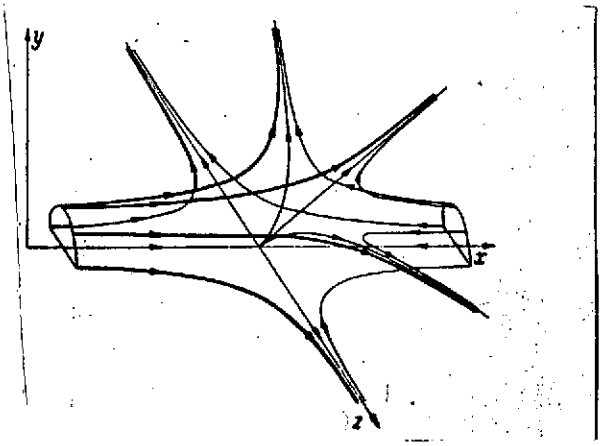

Fig. 7. Point of divergence of strong divergence (Case 9).

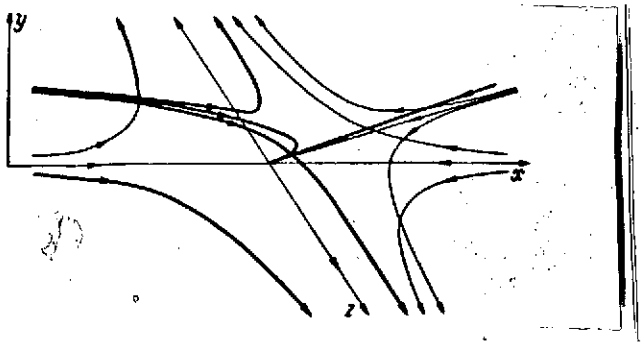

Fig. 8. Point of divergence without separation (Case 11.)

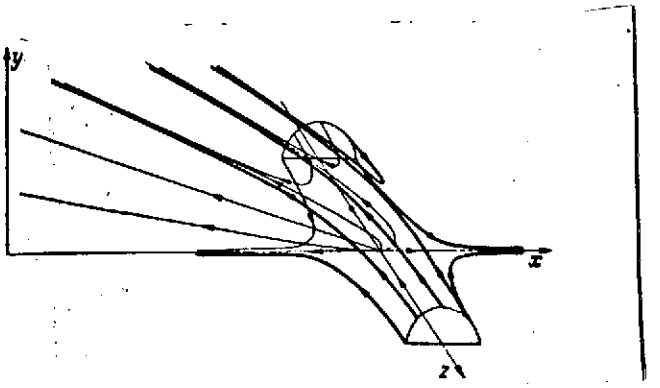

Fig. 9. Point of convergence with separation despite negative θ (Case 1).

Figure 8 (Case 11) is of special interest. In this case, /362
divergence is assumed to be so strong that flow -- although flowing toward the flow plane of separation $x = y\cot\theta$ -- does not separate. On the "separation line" $x = \cot\theta$, $z = 0$, the /364
flow, as in the entire x,y-plane, tends toward the center and continues in the outflow along the z-axis. The other flow lines approach the separation plane $x = y\cot\theta$ and the wall $y = 0$ simultaneously. We are, therefore, dealing solely with a "deflection" of a flow by a return-flow, but no longer with a separation from the wall. $\tau = \sigma = 0$, therefore, is merely necessary but not sufficient for separation in the strict sense of the word.

Figure 9 (Case 1) occupies a special position and at the same time forms a counterpart to Fig. 8. An especially strong convergence is assumed at $p_x < 0$ (pressure drop!) and $\tau_x > 0$, as would be commensurate with rejoining of a two-dimensional flow. In spite of this, the flow separates on account of the strong convergence toward the x,y-plane.

5. The Symmetrical, Three-Dimensional Point of Separation

In the general case, too, $\tau = 0$, $\sigma = 0$ is the necessary prerequisite for a flow separation line. By generalization of (21), the following results: /365

$$\left.\begin{aligned}\frac{dx}{dy} &= \frac{2\tau_x \cdot x + p_x \cdot y + 2\tau_z \cdot z}{-(\tau_x + \sigma_x)\,y};\\ \frac{dz}{dy} &= \frac{2\sigma_x \cdot x + p_z \cdot y + 2\sigma_z \cdot z}{-(\tau_z + \sigma_z)\,y}.\end{aligned}\right\} \qquad (27)$$

The flow separation line

$$\frac{x}{y} = \cot\theta, \qquad \frac{z}{y} = \cot\varphi \cdot \cot\theta \qquad (28)$$

is given by

$$\left.\begin{aligned} \cot\vartheta &= \frac{2\,p_z\,\tau_z - p_x\,(\tau_x + 3\,\sigma_z)}{(3\,\tau_x + \sigma_z)(\tau_x + 3\,\sigma_z) - 4\,\sigma_x\,\tau_z}\,; \\ \cot\psi\cdot\cot\vartheta &= \frac{2\,p_x\,\sigma_x - p_z\,(3\,\tau_x + \sigma_z)}{(3\,\tau_x + \sigma_z)(\tau_x + 3\,\sigma_z) - 4\,\sigma_x\,\tau_z}\,. \end{aligned}\right\} \qquad (29)$$

Analogous to (23), by means of

$$\xi = x - y\cot\vartheta;\quad \eta = y;\quad \zeta = z - \cot\psi\cot\vartheta \qquad (30)$$

the flow separation line to the axis of an affine coordinate system is formed, and instead of (27), the following results:

$$\left.\begin{aligned} -(\tau_x + \sigma_z)\,y\cdot\frac{d\,\xi}{d\,y} &= 2\,\tau_x\cdot\xi + 2\,\tau_z\cdot\zeta\,; \\ -(\tau_x + \sigma_z)\,y\cdot\frac{d\,\zeta}{d\,y} &= 2\,\sigma_x\cdot\xi + 2\,\sigma_z\cdot\zeta\,. \end{aligned}\right\} \qquad (31)$$

Since pressure no longer occurs explicitly, only the derivatives of the wall shearing stresses are pertinent. The following solution for (31) is obtained by using

$$\lambda_{1,2} = \frac{-(\tau_x + \sigma_z) \pm \sqrt{(\sigma_z - \tau_x)^2 + 4\,\tau_z\,\sigma_x}}{\tau_x + \sigma_z}$$

and two arbitrary constants A_1, B_2:

$$\left.\begin{aligned} \xi &= A_1\,y^{\lambda_1} + B_2\cdot\frac{1}{2\,\sigma_x}\,(\tau_x - \sigma_z + \sqrt{(\sigma_z - \tau_x)^2 + 4\,\tau_z\,\sigma_x})\,y^{\lambda_2}\,; \\ \zeta &= -A_1\cdot\frac{1}{2\,\tau_z}\,(\tau_x - \sigma_z + \sqrt{(\sigma_z - \tau_x)^2 + 4\,\tau_z\,\sigma_x})\,y^{\lambda_1} + B_2\,y^{\lambda_2}\,; \end{aligned}\right\} \quad \lambda_1 \neq \lambda_2 \qquad (32)$$

$$\left.\begin{aligned} \xi &= \frac{A_1}{y} + \frac{2\,\tau_z}{\sigma_z - \tau_x}\cdot\frac{B_2}{y}\ln y\,; \\ \zeta &= -\frac{2\,\sigma_z\,A_1 + (\sigma_z + \tau_x)\cdot B_2}{\sigma_z - \tau_x}\cdot\frac{1}{y} + \frac{B_2}{y}\ln y\,; \end{aligned}\right\} \quad \lambda_1 = \lambda_2 = -1 \qquad (32')$$

where, in case of complex λ_1, λ_2, the real component must be taken from the right side.

In place of (32), with arbitrary A_3, B_4, it is also possible to write

$$\left.\begin{aligned} \xi - \frac{1}{2\sigma_x}(\tau_z - \sigma_z + \sqrt{(\sigma_z - \tau_z)^2 + 4\tau_z\sigma_x})\,\zeta &= A_3\, y^{\lambda_1}, \\ \frac{1}{2\tau_z}(\tau_z - \sigma_z + \sqrt{(\sigma_z - \tau_z)^2 + 4\tau_z\sigma_x})\cdot\xi + \zeta &= B_3\, y^{\lambda_2}. \end{aligned}\right\} \qquad (33)$$

A comparison with (25) shows that for /366

$$(\sigma_z - \tau_x)^2 + 4\tau_z\sigma_x > 0 \quad \text{(i.e. } \lambda_1, \lambda_2 \text{ real)} \qquad (34)$$

the general case can be reduced to the symmetrical point of separation by means of an additional (real) affine transformation. Premise (34) yields the following for the wall flow lines:

$$\{2\sigma_x \cdot x - [\tau_x - \sigma_z + \sqrt{(\sigma_z - \tau_z)^2 + 4\tau_z\sigma_x}]\cdot z\}\cdot\{[\tau_x - \sigma_z + \sqrt{(\sigma_z - \tau_z)^2 + 4\tau_z\sigma_x}]\cdot x + 2\tau_z \cdot z\}^{-\frac{\lambda_1}{\lambda_2}} = \text{const.} \qquad (35)$$

There is a point of convergence for $\lambda_1/\lambda_2 > 0$, i.e., $\tau_x\sigma_z > \tau_z\sigma_x$, a point of divergence for $\lambda_1/\lambda_2 < 0$, i.e. $\tau_x\sigma_z < \tau_z\sigma_x$. As already occurred in the symmetrical case, there results such strong divergence for $\tau_x + \sigma_z < 0$, that no separation takes place any more.

However, if instead of (34), $(\sigma_z - \tau_x)^2 + 4\tau_z\sigma_x < 0$, then it is appropriate to put

$$(\tau_z + \sigma_z)\,\lambda'' = \sqrt{-(\sigma_z - \tau_z)^2 - 4\tau_z\sigma_x} \qquad (36)$$

and the use of a,α as arbitrary constants yields the solution in the form

$$\left.\begin{aligned} \xi + \frac{\sigma_z - \tau_x}{2\sigma_x}\cdot\zeta &= \frac{a}{y}\cdot\sin(\lambda''\ln y - \alpha); \\ \frac{\sigma_z + \tau_x}{2\sigma_x}\cdot\lambda''\cdot\zeta &= \frac{a}{y}\cos(\lambda''\ln y - \alpha). \end{aligned}\right\} \qquad (37)$$

One obtains as the equation of the wall flow lines

$$\operatorname{arctg}\frac{2\sigma_x\cdot x+(\sigma_z-\tau_x)\cdot z}{(\sigma_z+\tau_x)\lambda''\cdot z}=$$
$$=-\lambda''\ln\sqrt{\left(x+\frac{\sigma_z-\tau_x}{2\sigma_x}z\right)^2+\left(\frac{\sigma_z+\tau_x}{2\sigma_x}\lambda''z\right)^2}+\text{const.} \qquad (38)$$

In the present case, we are dealing with a vortex[2]; λ'' determines the strength of the vortex. Figures 10 and 11 show a vortex of average convergence and average or weak strength. It is typical that a line of separation as division into onflow and return-flow region is lacking. The wall flow lines end at the origin only after an infinite number of turns and from there continue in the separation lines. It is a matter of definition whether such an attitude can still be termed a separation.

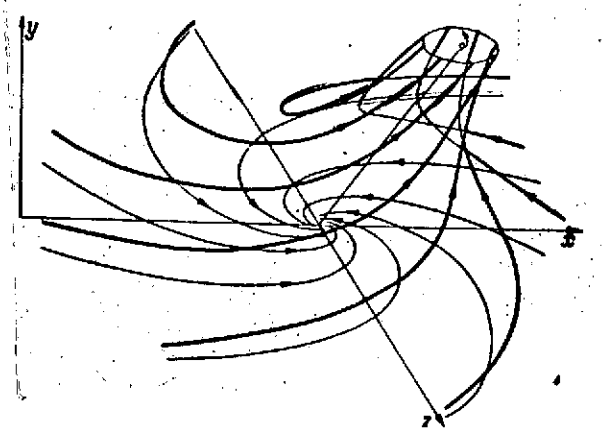

Fig. 10. Vortex of average strength and convergence.

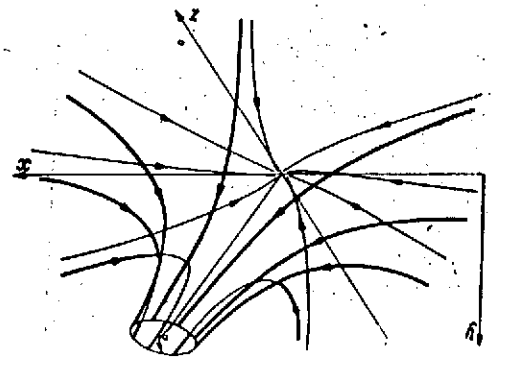

Fig. 11. Vortex of weak strength and average convergence.

The point of separation of average convergence (Fig. 2) shows a special tendency toward the formation of a vortex. It is suspected that the eddies, which originate at the forward end of bodies of rotation and wings of small aspect ratio of large pitch, issue from a vortex.

[2]The vortex was predicted by R. Legendre [6].

The exceptional case of $\lambda_1 = \lambda_2 = -1$ offers nothing that is essentially new. /367

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